

# Hyperfine structure of $S$ -states in muonic deuterium

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On the basis of quasipotential method in quantum electrodynamics we calculate corrections of order  $\alpha^5$  and  $\alpha^6$  to hyperfine structure of  $S$ -wave energy levels of muonic deuterium. Relativistic corrections, effects of vacuum polarization in first, second and third orders of perturbation theory, nuclear structure and recoil corrections are taken into account. The obtained numerical values of hyperfine splitting  $\Delta E^{hfs}(1S) = 50.2814$  meV ( $1S$  state) and  $\Delta E^{hfs}(2S) = 6.2804$  meV ( $2S$  state) represent reliable estimate for a comparison with forthcoming experimental data of CREMA collaboration. The hyperfine structure interval  $\Delta_{12} = 8\Delta E^{hfs}(2S) - \Delta E^{hfs}(1S) = -0.0379$  meV can be used for precision check of quantum electrodynamics predictions for muonic deuterium.

## I. INTRODUCTION

In last years a considerable interest in the investigation of fine and hyperfine energy structure of simple atoms is related with light muonic atoms: muonic hydrogen, muonic deuterium, ions of muonic helium. This is caused by essential progress achieved by experimental collaboration CREMA (Charge Radius Experiment with Muonic Atoms) in the study of such muonic atoms [1, 2]. Thus, for example, in the measurement of transition frequency  $2P_{3/2}^{F=2} - 2S_{1/2}^{F=1}$  there was obtained more precise value of proton charge radius  $r_p = 0.84087(39)$  fm. The measurement of transition frequency  $2P_{3/2}^{F=1} - 2S_{1/2}^{F=0}$  for singlet  $2S$ -state allowed to find hyperfine splitting (HFS) of  $2S$  energy level in muonic hydrogen and the value of the Zemach radius  $r_Z = 1.082(37)$  fm and magnetic radius  $r_M = 0.87(6)$  fm. Analogous measurements for muonic deuterium were also completed and planned for a publication. It is necessary to point out that the experiments of CREMA collaboration propose one important task to improve by order of magnitude the value of charge radii in these simple atoms (proton, deuteron, helion and  $\alpha$ -particle) which enter to theoretical expressions for different fine structure intervals. For successful implementation of this program theoretical calculations of different order corrections to fine and hyperfine structure

of muonic atoms have a significant importance. [3–8]. Of special note are nuclear structure corrections which can be responsible for the solution of the proton radius puzzle [1, 9].

Theoretical investigations of the energy levels of light muonic atoms were carried out many years ago in [3–6, 8] (other references can be found in [8]) on the basis of the Dirac equation and nonrelativistic approach by perturbation theory in quantum electrodynamics. An experimental activity in last years generates a need to analyze again previous calculations in order to obtain reliably basic energy intervals: the Lamb shift ( $2P_{1/2} - 2S_{1/2}$ ), hyperfine structure of  $2S$ -state, fine structure ( $2P_{1/2} - 2P_{3/2}$ ), which could be measured in CREMA experiments in the first place. The aim of our work consists in performing new investigation of contributions  $\alpha^5$  and  $\alpha^6$  in hyperfine structure of muonic deuterium which are determined by effects of the vacuum polarization, recoil, relativistic and deuteron structure corrections. Modern numerical values of fundamental physical constants are taken from [10]: the electron mass  $m_e = 0.510998928(11) \cdot 10^{-3}$  GeV, the muon mass  $m_\mu = 0.1056583715(35)$  GeV, fine structure constant  $\alpha^{-1} = 137.035999074(44)$ , the proton mass  $m_p = 0.938272046(21)$  GeV, the deuteron mass  $m_d = 1.875612859(41)$  GeV, the deuteron magnetic moment  $\mu_d = 0.8574382308(72)$  in nuclear magnetons, muon anomalous magnetic moment  $a_\mu = 1.16592091(63) \cdot 10^{-3}$ .

In our calculation we use the quasipotential method in quantum electrodynamics as applied to the particle bound states [11], where two-particle bound state is described by the Schrödinger equation. The basic contribution to muon-deuteron interaction operator in  $S$ -state is determined by the Breit Hamiltonian [12]:

$$H_B = H_0 + \Delta V_B^{fs} + \Delta V_B^{hfs}, \quad H_0 = \frac{\mathbf{p}^2}{2\mu} - \frac{Z\alpha}{r}, \quad (1)$$

$$\Delta V_B^{fs} = -\frac{\mathbf{p}^4}{8m_1^3} - \frac{\mathbf{p}^4}{8m_2^3} + \frac{\pi Z\alpha}{2} \left( \frac{1}{m_1^2} + \frac{\delta_I}{m_2^2} \right) \delta(\mathbf{r}) - \frac{Z\alpha}{2m_1 m_2 r} \left( \mathbf{p}^2 + \frac{\mathbf{r}(\mathbf{r}\mathbf{p})\mathbf{p}}{r^2} \right), \quad (2)$$

$$\Delta V_B^{hfs}(r) = \frac{2\pi\alpha}{3m_1 m_p} g_d g_\mu (\mathbf{s}_1 \mathbf{s}_2) \delta(\mathbf{r}), \quad (3)$$

where  $m_1, m_2$  are the muon and deuteron masses respectively,  $m_p$  is the proton mass,  $g_d, g_\mu$  are gyromagnetic factors of the deuteron and muon. The deuteron factor  $\delta_I = 0$ , because we use the following definition of the deuteron charge radius  $r_d^2 = -6 \frac{dF_C}{dQ^2}|_{Q^2=0}$  [13, 14]. Then the basic contribution to the hyperfine splitting of  $S$ -wave levels (the Fermi energy) is given by spin-spin interaction part of the potential (3). Averaging (3) over the Coulomb wave functions of  $1S$ - and  $2S$ -states

$$\psi_{100}(r) = \frac{W^{3/2}}{\sqrt{\pi}} e^{-Wr}, \quad W = \mu Z\alpha, \quad (4)$$

$$\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-Wr/2} \left( 1 - \frac{Wr}{2} \right), \quad (5)$$

we obtain the following results for the leading order contribution to hyperfine splitting:

$$E_F(nS) = \frac{2\mu^3 \alpha^4 \mu_d}{m_1 m_p n^3} = \begin{cases} 1S : 49.0875 \text{ meV} \\ 2S : 6.1359 \text{ meV} \end{cases}, \quad (6)$$

The Fermi energy (6) does not contain a contribution of muon anomalous magnetic moment (AMM). The muon AMM correction to hyperfine splitting can be presented separately taking experimental value of muon AMM [10]:

$$\Delta E_{a_\mu}^{hfs}(nS) = a_\mu E_F(nS) = \begin{cases} 1S : 0.0572 \text{ meV} \\ 2S : 0.0072 \text{ meV} \end{cases} \quad (7)$$

Numerical value of relativistic correction of order  $\alpha^6$  to HFS can be obtained by means of known analytical expression from [8, 15]:

$$\Delta E_{rel}^{hfs}(nS) = \begin{cases} \frac{3}{2}(Z\alpha)^2 E_F(1S) \\ \frac{17}{8}(Z\alpha)^2 E_F(2S) \end{cases} = \begin{cases} 1S : 0.0039 \text{ meV} \\ 2S : 0.0007 \text{ meV} \end{cases} \quad (8)$$

In what follows we investigate other important contributions to HFS of  $S$ -wave energy levels in order to obtain reliable total result. Numerical values of different corrections are presented for definiteness with the accuracy  $10^{-4}$  meV.

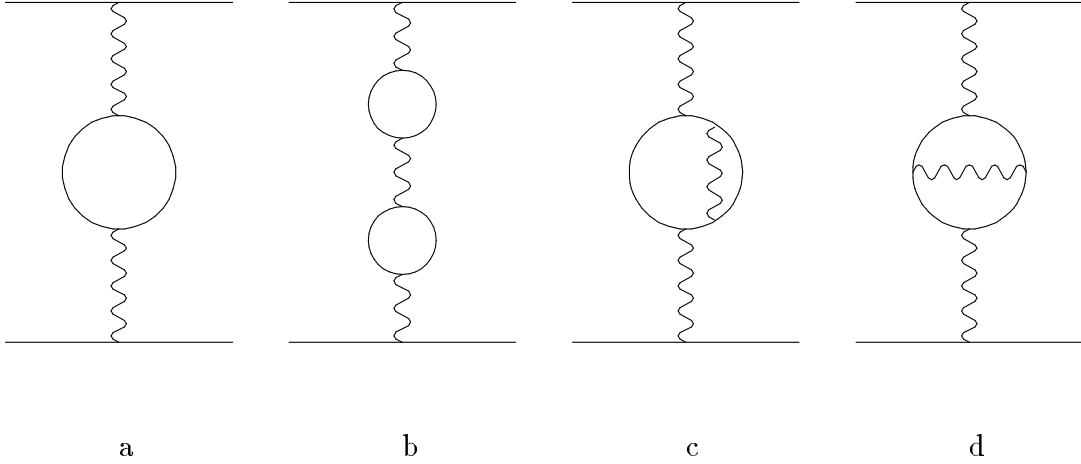


FIG. 1: Effects of one- and two-loop vacuum polarization in one-photon interaction.

## II. EFFECTS OF ONE- AND TWO-LOOP VACUUM POLARIZATION IN FIRST ORDER PERTURBATION THEORY

First of all, we should analyze a contribution of one-loop vacuum polarization to the potential, which is determined in coordinate representation as follows [4, 12]:

$$\Delta V_{1\gamma,VP}^{hfs}(r) = \frac{4\alpha\mu_d(1+a_\mu)}{3m_1m_p}(\mathbf{s}_1\mathbf{s}_2)\frac{\alpha}{3\pi}\int_1^\infty \rho(s)ds \left( \pi\delta(\mathbf{r}) - \frac{m_e^2\xi^2}{r}e^{-2m_e\xi r} \right), \quad (9)$$

where spectral function  $\rho(\xi) = \sqrt{\xi^2 - 1}(2\xi^2 + 1)/\xi^4$ . For its derivation a replacement in photon propagator is used:

$$\frac{1}{k^2} \rightarrow \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \frac{1}{k^2 + 4m_e^2\xi^2}. \quad (10)$$

We also preserve a factor with muon AMM that leads to the accounting effectively a correction of order  $\alpha^6$ . Averaging (9) over wave functions (4) and (5), we obtain the contribution of order  $\alpha^5$  to hyperfine structure of  $1S$ - and  $2S$ -states:

$$\Delta E_{1\gamma,VP}^{hfs}(1S) = \frac{2\mu^3\alpha^5\mu_d(1+a_\mu)}{3m_1m_p\pi} \int_1^\infty \rho(\xi)d\xi \left[ 1 - \frac{m_e^2\xi^2}{W^2} \int_0^\infty xdx e^{-x(1+\frac{m_e\xi}{W})} \right] = 0.1039 \text{ meV}, \quad (11)$$

$$\Delta E_{1\gamma,VP}^{hfs}(2S) = \frac{\mu^3\alpha^5\mu_d(1+a_\mu)}{3m_1m_p\pi} \int_1^\infty \rho(\xi)d\xi \times \left[ 1 - \frac{4m_e^2\xi^2}{W^2} \int_0^\infty x \left(1 - \frac{x}{2}\right)^2 dx e^{-x(1+\frac{2m_e\xi}{W})} \right] = 0.0134 \text{ meV}. \quad (12)$$

Changing the electron mass  $m_e$  to muon mass  $m_1$  in (11) and (12), the muon vacuum polarization contribution to HFS can be found: 0.0009 meV ( $1S$ ), 0.0001 meV ( $2S$ ). It has higher order  $\alpha^6$  because the ratio  $W/m_1 \ll 1$  and is included in Table I in corresponding line. The same order  $\alpha^6$  contribution is given also by two-loop vacuum polarization diagrams (see Fig. 1(b,c,d) (the Källen and Sabry potential [16])). In order to obtain the interaction operator for the amplitude with two sequential loops (Fig. 1(b)), it is necessary to use twice a replacement (10). Thus in coordinate space a potential takes a form:

$$\Delta V_{1\gamma,VP-VP}^{hfs}(r) = \frac{8\pi\alpha\mu_d(1+a_\mu)}{3m_1m_p} (\mathbf{s}_1\mathbf{s}_2) \left(\frac{\alpha}{3\pi}\right)^2 \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \times \left[ \delta(\mathbf{r}) - \frac{m_e^2}{\pi r(\eta^2 - \xi^2)} (\eta^4 e^{-2m_e\eta r} - \xi^4 e^{-2m_e\xi r}) \right]. \quad (13)$$

Corresponding correction to HFS of levels  $1S$  and  $2S$  can be presented firstly in integral form over coordinate  $r$  and spectral parameters  $\xi$  and  $\eta$ . After that the integration over  $r$  can be done analytically and two other integrations numerically with the use a system Mathematica. Two-loop vacuum polarization correction of order  $\alpha^6$  in Fig. 1(c,d) can be calculated similarly. In this case a potential of muon-deuteron interaction is determined by more complicated expression

$$\Delta V_{1\gamma,2-loop VP}^{hfs}(r) = \frac{8\alpha^3\mu_d(1+a_\mu)}{9\pi^2m_1m_p} (\mathbf{s}_1\mathbf{s}_2) \int_0^1 \frac{f(v)dv}{1-v^2} \left[ \pi\delta(\mathbf{r}) - \frac{m_e^2}{r(1-v^2)} e^{-\frac{2m_er}{\sqrt{1-v^2}}} \right], \quad (14)$$

where two-loop spectral function

$$f(v) = v \left\{ (3-v^2)(1+v^2) \left[ Li_2 \left( -\frac{1-v}{1+v} \right) + 2Li_2 \left( \frac{1-v}{1+v} \right) + \frac{3}{2} \ln \frac{1+v}{1-v} \ln \frac{1+v}{2} - \ln \frac{1+v}{1-v} \ln v \right] \right. \\ \left. + \left[ \frac{11}{16}(3-v^2)(1+v^2) + \frac{v^4}{4} \right] \ln \frac{1+v}{1-v} + \left[ \frac{3}{2}v(3-v^2) \ln \frac{1-v^2}{4} - 2v(3-v^2) \ln v \right] + \frac{3}{8}v(5-3v^2) \right\}, \quad (15)$$

$Li_2(z)$  the Euler dilogarithm. Numerical corrections of the operator (14) to the energy spectrum are evaluated as in the case of (13). Summary corrections from potentials (13) and (14) are equal

$$\Delta E_{1\gamma,VP,VP}^{hfs}(nS) = \begin{cases} 1S : 0.0005 \text{ meV} \\ 2S : 0.00006 \text{ meV} \end{cases} \quad (16)$$

One-loop and two-loop contributions of order  $\alpha^5$  and  $\alpha^6$  to HFS should be considered also in second order perturbation theory.

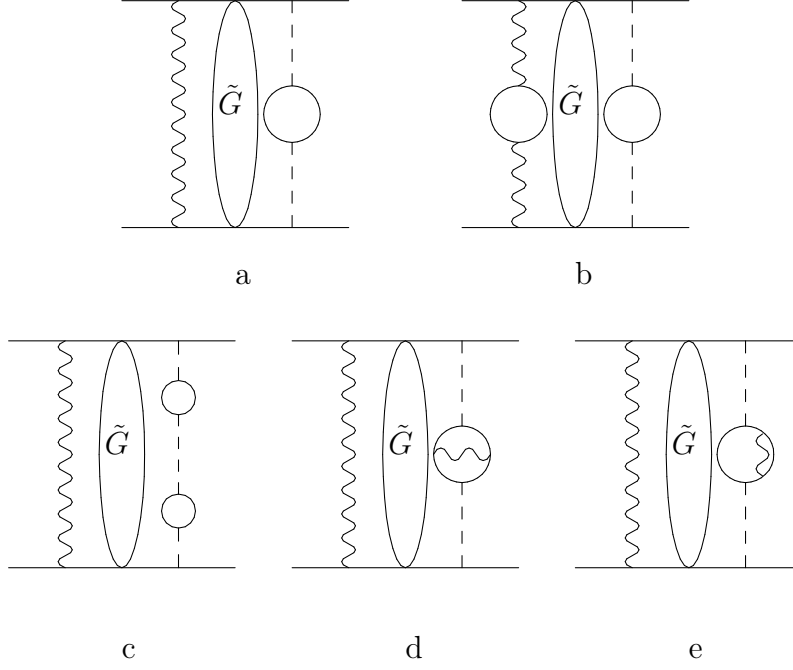


FIG. 2: Effects of one- and two-loop vacuum polarization in second order perturbation theory. Dashed line denotes the Coulomb photon.  $\tilde{G}$  is the reduced Coulomb Green function.

### III. EFFECTS OF ONE- AND TWO-LOOP VACUUM POLARIZATION IN SECOND AND THIRD ORDER PERTURBATION THEORY

The second order perturbation theory (PT) corrections to the energy spectrum are determined by the reduced Coulomb Green's function  $\tilde{G}$ , which has the following partial expansion:

$$\tilde{G}_n(\mathbf{r}, \mathbf{r}') = \sum_{l,m} \tilde{g}_{nl}(r, r') Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}'). \quad (17)$$

The radial function  $\tilde{g}_{nl}(r, r')$  was obtained in [17] in the form of the Sturm expansion in the Laguerre polynomials. The main contribution of the electron vacuum polarization to HFS in second order PT has the form (see Fig. 2(a)):

$$\Delta E_{SOPT \text{ VP } 1}^{hfs} = 2 \langle \psi | \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V_B^{hfs} | \psi \rangle, \quad (18)$$

where the modified Coulomb potential

$$\Delta V_{VP}^C(r) = \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \left( -\frac{Z\alpha}{r} \right) e^{-2m_e \xi r}. \quad (19)$$

Since  $\Delta V_B^{hfs}(r)$  is proportional to  $\delta(\mathbf{r})$ , it is necessary to use the reduced Coulomb Green's function with one zero argument. For this case it was obtained on the basis of the Hostler representation after a subtraction of the pole term and has the form [17]:

$$\tilde{G}_{1S}(\mathbf{r}, 0) = \frac{Z\alpha\mu^2}{4\pi} \frac{e^{-x}}{x} g_{1S}(x), \quad g_{1S}(x) = [4x(\ln 2x + C) + 4x^2 - 10x - 2], \quad (20)$$

$$\tilde{G}_{2S}(\mathbf{r}, 0) = -\frac{Z\alpha\mu^2}{4\pi} \frac{e^{-x/2}}{2x} g_{2S}(x), g_{2S}(x) = [4x(x-2)(\ln x + C) + x^3 - 13x^2 + 6x + 4], \quad (21)$$

where  $C = 0.5772\dots$  is the Euler constant and  $x = Wr$ . As a result necessary corrections to HFS of  $(\mu d)$  can be presented as follows:

$$\Delta E_{VP\ 1}^{hfs}(1S) = -E_F(1S) \frac{\alpha}{3\pi} (1 + a_\mu) \int_1^\infty \rho(\xi) d\xi \int_0^\infty e^{-x(1+\frac{m_e\xi}{W})} g_{1S}(x) dx = 0.2056 \text{ meV}, \quad (22)$$

$$\Delta E_{VP\ 1}^{hfs}(2S) = E_F(2S) \frac{\alpha}{3\pi} (1 + a_\mu) \int_1^\infty \rho(\xi) d\xi \int_0^\infty e^{-x(1+\frac{2m_e\xi}{W})} g_{2S}(x) (1 - \frac{x}{2}) dx = 0.0207 \text{ meV}. \quad (23)$$

The factor  $(1+a_\mu)$  is included in (22) and (23), therefore these expressions contain corrections of order  $\alpha^5$  and  $\alpha^6$ . Changing  $m_e \rightarrow m_1$  in (22)-(23) we calculate one-loop muon vacuum polarization contribution in second order PT of order  $\alpha^6$ : 0.0009 meV (1S), 0.0001 meV (2S). It is included also in Table I together with similar contribution in first order PT. Two-loop corrections in Fig. 2(b,c,d,e) are of order  $\alpha^6$ . Let us consider first contribution which is related with potentials (9) and (19), reduced Coulomb Green's functions (20), (21) and reduced Coulomb Green's function with nonzero arguments. General structure of this contribution takes the form:

$$\Delta E_{SOPT\ VP\ 2}^{hfs} = 2 \langle \psi | \Delta V_{1\gamma, VP}^{hfs} \cdot \tilde{G} \cdot \Delta V_{VP}^C | \psi \rangle. \quad (24)$$

The convenient representation for reduced Coulomb Green's function with nonzero arguments was obtained in [17]:

$$\tilde{G}_{1S}(r, r') = -\frac{Z\alpha\mu^2}{\pi} e^{-(x_1+x_2)} g_{1S}(x_1, x_2), \quad (25)$$

$$g_{1S}(x_1, x_2) = \frac{1}{2x_<} - \ln 2x_> - \ln 2x_< + Ei(2x_<) + \frac{7}{2} - 2C - (x_1 + x_2) + \frac{1 - e^{2x_<}}{2x_<},$$

$$\tilde{G}_{2S}(r, r') = -\frac{Z\alpha\mu^2}{16\pi x_1 x_2} e^{-(x_1+x_2)} g_{2S}(x_1, x_2), \quad (26)$$

$$g_{2S}(x_1, x_2) = 8x_< - 4x_<^2 + 8x_> + 12x_<x_> - 26x_<^2x_> + 2x_<^3x_> - 4x_>^2 - 26x_<x_>^2 + 23x_<^2x_>^2 - \\ - x_<^3x_>^2 + 2x_<x_>^3 - x_<^2x_>^3 + 4e^x(1 - x_<)(x_> - 2)x_> + 4(x_< - 2)x_<(x_> - 2)x_> \times \\ \times [-2C + Ei(x_<) - \ln(x_<) - \ln(x_>)].$$

The substitution of (9), (19), (25) and (26) into (24) provides two contributions for each 1S and 2S level in integral form:

$$\Delta E_{VP\ 21}^{hfs}(1S) = -\frac{2\alpha^6\mu^3\mu_d(1+a_\mu)}{9\pi^2m_1m_p} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \int_0^\infty dx e^{-x(1+\frac{m_e\xi}{W})} g_{1S}(x), \quad (27)$$

$$\Delta E_{VP\ 22}^{hfs}(1S) = -\frac{4\alpha^6\mu^3\mu_d(1+a_\mu)m_e^2}{9\pi^2m_1m_pW^2} \int_1^\infty \rho(\xi) d\xi \times \\ \times \int_1^\infty \rho(\eta) \eta^2 d\eta \int_0^\infty x_1 dx_1 e^{-x_1(1+\frac{m_e\xi}{W})} \int_0^\infty x_2 dx_2 e^{-x_2(1+\frac{m_e\xi}{W})} g_{1S}(x_1, x_2), \quad (28)$$

$$\Delta E_{VP\ 21}^{hfs}(2S) = \frac{\alpha^6 \mu^3 \mu_d (1 + a_\mu)}{36\pi^2 m_1 m_p} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \int_0^\infty \left(1 - \frac{x}{2}\right) dx e^{-x(1 + \frac{2m_e \xi}{W})} g_{2S}(x), \quad (29)$$

$$\begin{aligned} \Delta E_{VP\ 22}^{hfs}(2S) = & -\frac{\alpha^6 \mu^3 \mu_d (1 + a_\mu) m_e^2}{18\pi^2 m_1 m_p W^2} \int_1^\infty \rho(\xi) d\xi \times \\ & \times \int_1^\infty \rho(\eta) \eta^2 d\eta \int_0^\infty \left(1 - \frac{x_1}{2}\right) dx_1 e^{-x_1(1 + \frac{2m_e \xi}{W})} \int_0^\infty \left(1 - \frac{x_2}{2}\right) dx_2 e^{-x_2(1 + \frac{2m_e \eta}{W})} g_{2S}(x_1, x_2). \end{aligned} \quad (30)$$

Separately, the contributions (27), (28) and (29),(30) are divergent but their sum is finite. Corresponding numerical values are presented in Table I. The contributions of two other diagrams to HFS can be calculated by means of (18), where the replacement of the potential (19) on the following potentials should be made [7]:

$$\Delta V_{VP-VP}^C(r) = \left(\frac{\alpha}{3\pi}\right)^2 \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \left(-\frac{Z\alpha}{r}\right) \frac{1}{\xi^2 - \eta^2} (\xi^2 e^{-2m_e \xi r} - \eta^2 e^{-2m_e \eta r}), \quad (31)$$

$$\Delta V_{2-loop\ VP}^C(r) = -\frac{2Z\alpha^3}{3\pi^2 r} \int_0^1 \frac{f(v) dv}{(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}. \quad (32)$$

Omitting further intermediate expressions we include in Table I numerical values of corrections from potentials (29) and (30). Besides vacuum polarization corrections there are a number of nuclear structure and recoil contributions playing essential role in hyperfine splitting.

In third order perturbation theory there is a correction of order  $\alpha^6$  to hyperfine splitting which is represented symbolically in Fig.3 (see [18]). The initial expression for it is the following:

$$\begin{aligned} \Delta E_{TOPT}^{hfs} = & \langle \psi_n | \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V^{hfs} \cdot \tilde{G} \cdot \Delta V_{VP}^C | \psi_n \rangle + 2 \langle \psi_n | \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V^{hfs} | \psi_n \rangle - \\ & - \langle \psi_n | \Delta V^{hfs} | \psi_n \rangle \langle \psi_n | \Delta V_{VP}^C \cdot \tilde{G} \cdot \tilde{G} \cdot \Delta V_{VP}^C | \psi_n \rangle - \\ & - 2 \langle \psi_n | \Delta V_{VP}^C | \psi_n \rangle \langle \psi_n | \Delta V_{VP}^C \cdot \tilde{G} \cdot \tilde{G} \cdot \Delta V^{hfs} | \psi_n \rangle. \end{aligned} \quad (33)$$

The integration over one coordinate in (33) is performed analytically, but second coordinate integration and two integrations over spectral parameters are calculated numerically. Numerically the contribution (33) is essentially smaller than other corrections of order  $\alpha^6$  (see Table I where it is written in separate line).

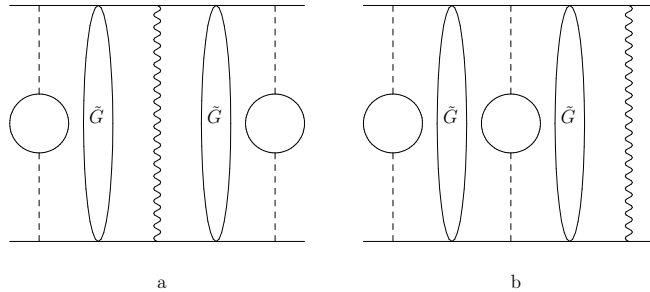


FIG. 3: Effects of vacuum polarization in third order perturbation theory. Dashed line denotes the Coulomb photon.  $\tilde{G}$  is the reduced Coulomb Green function.

#### IV. NUCLEAR STRUCTURE AND RECOIL CORRECTIONS.

The basic nuclear structure contribution to HFS of  $S$ -states is determined by two-photon exchange diagrams (see Fig. 4). The deuteron electromagnetic current parametrization takes the form:

$$J_d^\mu(p_2, q_2) = \varepsilon_\rho^*(q_2) \left\{ \frac{(p_2 + q_2)_\mu}{2m_2} g_{\rho\sigma} F_1(k^2) - \frac{(p_2 + q_2)_\mu}{2m_2} \frac{k_\rho k_\sigma}{2m_2^2} F_2(k^2) - \Sigma_{\rho\sigma}^{\mu\nu} \frac{k_\nu}{2m_2} F_3(k^2) \right\} \varepsilon_\sigma(p_2), \quad (34)$$

where  $p_2, q_2$  are four-momenta of the deuteron in initial and final states,  $k = q_2 - p_2$ . The deuteron polarization vector  $\varepsilon_\mu$  satisfies to the following conditions:

$$\varepsilon_\mu^*(\mathbf{k}, \lambda) \varepsilon^\mu(\mathbf{k}, \lambda') = -\delta_{\lambda\lambda'}, \quad k_\mu \varepsilon^\mu(\mathbf{k}, \lambda) = 0, \quad \sum_\lambda \varepsilon_\mu^*(\mathbf{k}, \lambda) \varepsilon_\nu(\mathbf{k}, \lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_2^2}. \quad (35)$$

The generator of infinitesimal Lorentz transformations is given by

$$\Sigma_{\rho\sigma}^{\mu\nu} = g_\rho^\mu g_\sigma^\nu - g_\sigma^\mu g_\rho^\nu. \quad (36)$$

The deuteron electromagnetic form factors  $F_i(k^2)$  are functions of the square of the photon four-momentum. They are related to the charge  $F_C$ , magnetic  $F_M$ , and quadrupole  $F_Q$  form factors for the deuteron by the equations

$$F_C = F_1 + \frac{2}{3}\eta [F_1 + (1 + \eta)F_2 - F_3], \quad F_M = F_3, \quad F_Q = F_1 + (1 + \eta)F_2 - F_3. \quad \eta = -\frac{k^2}{4m_2^2}. \quad (37)$$

The muon electromagnetic current has the form:

$$J_l^\mu(p_1, q_1) = \bar{u}(q_1) \left[ \frac{(p_1 + q_1)^\mu}{2m_1} - (1 + a_\mu) \sigma^{\mu\nu} \frac{k_\nu}{2m_1} \right] u(p_1), \quad (38)$$

where  $p_1, q_1$  are initial and final muon four-momenta,  $\sigma^{\mu\nu} = (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)/2$ . The amplitudes describing the virtual Compton scattering of a muon and a deuteron are defined by direct and crossed two-photon diagrams in the form [19]:

$$M_{\mu\nu}^{(l)} = \bar{u}(q_1) \left[ \gamma_\mu \frac{\hat{p}_1 + \hat{k} + m_1}{(p_1 + k)^2 - m_1^2} \gamma_\nu + \gamma_\nu \frac{\hat{p}_1 - \hat{k} + m_1}{(p_1 - k)^2 - m_1^2} \gamma_\mu \right] u(p_1), \quad (39)$$

$$M_{\mu\nu}^{(d)} = \varepsilon_\rho^*(q_2) \left[ \frac{(q_2 + p_2 - k)_\mu}{2m_2} g_{\rho\lambda} F_1 - \frac{(q_2 + p_2 - k)_\mu}{2m_2} \frac{k_\rho k_\lambda}{2m_2^2} F_2 - \Sigma_{\rho\lambda}^{\mu\alpha} \frac{k_\alpha}{2m_2} F_3 \right] \times \quad (40)$$

$$\frac{-g_{\lambda\omega} + \frac{(p_2 - k)_\lambda (p_2 - k)_\omega}{m_2^2}}{(p_2 - k)^2 - m_2^2} \left[ \frac{(p_2 + q_2 - k)_\nu}{2m_2} g_{\omega\sigma} F_1 - \frac{(p_2 + q_2 - k)_\nu}{2m_2} \frac{k_\omega k_\sigma}{2m_2^2} F_2 + \Sigma_{\omega\sigma}^{\nu\beta} \frac{k_\beta}{2m_2} F_3 \right] \varepsilon_\sigma(p_2).$$

To construct the quasipotential of hyperfine splitting we introduce special spin projection operators  $\hat{\pi}_{\mu,3/2}$  and  $\hat{\pi}_{\mu,1/2}$  on states with the muon-deuteron pair spin 3/2 and 1/2:

$$\hat{\Pi}_{\mu,3/2} = [u(p_1) \epsilon_\mu(p_2)]_{3/2} = \Psi_\mu(P), \quad \hat{\Pi}_{\mu,1/2} = \frac{i}{\sqrt{3}} \gamma_5 (\gamma_\mu - v_{1,\mu}) \Psi(P), \quad (41)$$



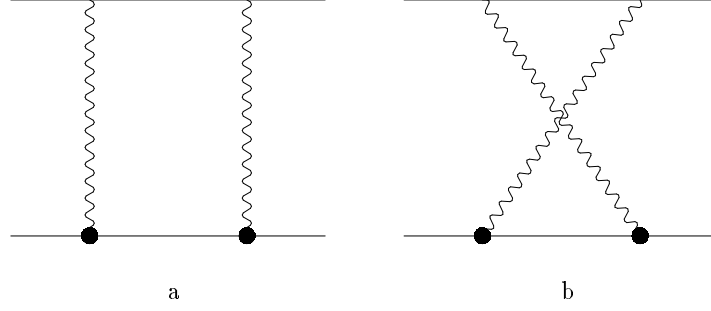


FIG. 4: Nuclear structure effects of order  $\alpha^5$ . The bold point denotes the deuteron vertex function.

$$\sum_{\lambda} \Psi_{\mu}^{\lambda}(P) \bar{\Psi}_{\nu}^{\lambda}(P) = -\frac{\hat{v}_1 + 1}{2} \left( g_{\mu\nu} - \frac{1}{3} \gamma_{\mu} \gamma_{\nu} - \frac{2}{3} v_{1,\mu} v_{1,\nu} + \frac{1}{3} (v_{1,\mu} \gamma_{\nu} - v_{1,\nu} \gamma_{\mu}) \right), \quad (42)$$

where the spin-vector  $\Psi_{\mu}(P)$  and spinor  $\Psi(P)$  describe bound states of the muon and deuteron with spins  $3/2$  and  $1/2$ ,  $v_{1,\mu} = P_{\mu}/M$ ,  $P = p_1 + p_2$ ,  $M = m_1 + m_2$ . Multiplying amplitudes (39) and (40) and introducing projection operators (41), we obtain by means of the package Form [20] the expression for HFS part of the potential of two-photon interaction in the Coulomb gauge for exchanged photons [19]:

$$V_{2\gamma, str}^{hfs} = (Z\alpha)^2 \int \frac{id^4k}{\pi^2} \frac{1}{(k^2)^2} \frac{1}{k^4 - 4k_0^2 m_1^2} \frac{1}{k^4 - 4k_0^2 m_2^2} \times \quad (43)$$

$$\left\{ 2F_1 F_3 k^6 \left( \frac{k^2}{m_2^2} - \frac{\mathbf{k}^2}{m_2^2} - 4 \right) + 2F_2 F_3 \frac{k^4}{m_2^2} \left( 4k_0^4 + \mathbf{k}^4 - 4\mathbf{k}^2 k_0^2 - \frac{k^6}{m_2^2} \right) + 2F_3^2 k^2 \mathbf{k}^2 \left( k_0^2 + \frac{k^4}{m_2^2} \right) \right\}.$$

The infrared divergence in (43) at  $k \rightarrow 0$  is related with a term  $\sim F_1 F_3 k^2$ . It can be eliminated by means of iteration term of the quasipotential:

$$\Delta V_{iter}^{hfs} = [V_{1\gamma} \times G^f \times V_{1\gamma}]^{HFS} = E_F \frac{16\mu\alpha}{3\pi n^3} (\mathbf{S}_1 \mathbf{S}_2) \int_0^{\infty} \frac{dk}{k^2} F_1 F_3. \quad (44)$$

The angle integration in (43) in the Euclidean momentum space can be carried out analytically. As a result the contribution of two-photon exchange amplitudes to HFS of  $S$ -levels can be written in the form of one-dimensional integral:

$$E_{2\gamma}^{hfs} = E_F \alpha \int_0^{\infty} \frac{dk}{k^2} V_{2\gamma}(k) = \quad (45)$$

$$= \frac{E_F \alpha}{16\pi m_1^3 m_2^5 (m_1^2 - m_2^2)} \int_0^{\infty} \frac{dk}{k^2} \left\{ 4m_1^2 m_2^2 F_1 F_3 \left[ k^5 (m_2^2 - m_1^2) + 8k^2 m_1^2 m_2^2 (h_2 - h_1) + \right. \right.$$

$$16m_1^2 m_2^4 (h_2 - h_1) - 32m_1^2 m_2^4 (m_2 - m_1) + k^4 (m_1^2 h_2 - m_2^2 h_1) \left. \right] +$$

$$+ F_2 F_3 k^2 \left[ k^5 (m_2^4 - m_1^4) + 6k^3 m_1^2 m_2^2 (m_1^2 - m_2^2) + 8k^2 m_1^2 m_2^2 (m_1^2 (h_2 - 2h_1) + m_2^2 h_1) + \right.$$

$$16m_1^4 m_2^4 (h_2 - h_1) + k^4 (m_1^4 h_2 - m_2^4 h_1) \left. \right] + F_3^2 k^2 m_2^2 \left[ k^3 (m_1^2 - m_2^2) (5m_1^2 + m_2^2) + \right.$$

$$\left. k^2 (-5m_1^4 h_2 + m_2^4 h_1 + 4m_1^2 m_2^2 h_1) + 6km_1^2 m_2^2 (m_1^2 - m_2^2) \right] \left. \right\},$$

TABLE I: Hyperfine structure of  $S$ -states of muonic deuterium.

Contribution to hyperfine splitting	$1S$ , meV	$2S$ , meV	Reference, equation
Contribution of order $\alpha^4$ , the Fermi energy	49.0875	6.1359	(6), [3]
Muon AMM contribution	0.0572	0.0072	(7), [3]
Relativistic correction of order $\alpha^6$	0.0039	0.0007	(8), [15],[3]
Vacuum polarization contribution of order $\alpha^5$	0.3095	0.0341	(11)-(12),(22)-(23),[3]
Vacuum polarization contribution of order $\alpha^6$	0.0048	0.0005	(16),(27)-(32)
Vacuum polarization contribution of order $\alpha^6$ in third order PT	0.0005	0.00004	(33)
Nuclear structure correction of order $\alpha^5$	-0.9305	-0.1163	(46)
Nuclear structure and vacuum polarization correction of order $\alpha^6$	0.0152	0.0019	(47)
Nuclear structure and muon vacuum polarization correction of order $\alpha^6$	0.0015	0.0002	(48)
Hadron vacuum polarization contribution of order $\alpha^6$	0.0018	0.0002	(50), [22]
Nuclear structure correction of order $\alpha^6$ in $1\gamma$ interaction	0.0082	0.0008	(55)
Nuclear structure correction in second order perturbation theory	-0.0555	-0.0069	(58)-(59)
Radiative nuclear finite size correction of order $\alpha^6$	-0.0039	-0.0005	(71)-(74)
Deuteron polarizability contribution of order $\alpha^5$	1.6972	0.2121	[5]
Deuteron internal polarizability contribution of order $\alpha^5$	0.0840	0.0105	[19]
Weak interaction contribution	0	0	[8, 30]
Summary contribution	50.2814	6.2804	

where the subtraction of iteration term of the potential (44) is performed,  $h_{1,2} = \sqrt{k^2 + 4m_{1,2}^2}$ . Moreover, we remove a factor  $F_3(0) = m_2\mu_d/Zm_p$  from the form factor  $F_3$  in (45) as in (44). Numerical integration in (45) performed with the use of known parametrization for the deuteron form factors [21] gives the following result:

$$\Delta E_{2\gamma, str}^{hfs}(nS) = \begin{cases} 1S : & -0.9305 \text{ meV} \\ 2S : & -0.1163 \text{ meV} \end{cases} \quad (46)$$

An expression (45) can be used for the evaluation of nuclear structure and vacuum polarization corrections shown in Fig. 5. A change of the potential in this case as compared with (45) can be derived after the replacement (10) and insertion of the factor 2. Corresponding

contribution to HFS of  $S$ -levels is represented in the form:

$$E_{2\gamma,VP}^{hfs} = -E_F \frac{2\alpha^2}{3\pi} \int_1^\infty \rho(\xi) d\xi \int_0^\infty V_{2\gamma}(k) \frac{dk}{k^2 + 4m_e^2 \xi^2} = \begin{cases} 1S : 0.0152 \text{ meV} \\ 2S : 0.0019 \text{ meV} \end{cases} \quad (47)$$

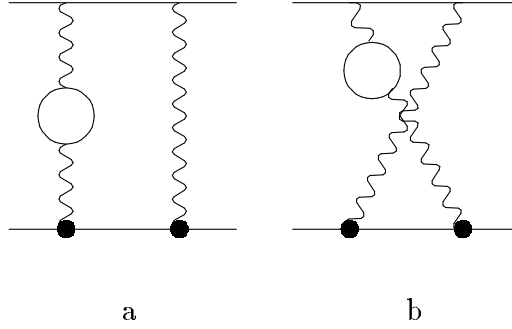


FIG. 5: Two photon exchange amplitudes accounting for effects of vacuum polarization and nuclear structure. The wavy line denotes the photon. The bold point denotes the deuteron vertex function.

The muon vacuum polarization contribution is also calculated by formula (47) after a replacement  $m_e \rightarrow m_1$ :

$$E_{2\gamma,MVP}^{hfs} = \begin{cases} 1S : 0.0015 \text{ meV} \\ 2S : 0.0002 \text{ meV} \end{cases} \quad (48)$$

To increase the calculation accuracy we consider hadron vacuum polarization (HVP) contribution. A potential  $V_{2\gamma}$  accounting for nuclear structure corrections is applicable for this aim. A replacement in photon propagator for HVP correction takes the form

$$\frac{1}{k^2} \rightarrow \left(\frac{\alpha}{\pi}\right) \int_{s_{th}}^\infty \frac{\rho_{had}(s) ds}{k^2 + s}, \quad (49)$$

and leads to the following expression for hadron vacuum polarization correction:

$$E_{2\gamma,HVP}^{hfs} = -E_F \frac{2\alpha^2}{\pi} \int_1^\infty \rho_{had}(s) ds \int_0^\infty V_{2\gamma}(k) \frac{dk}{k^2 + s}. \quad (50)$$

The basic contribution to hadron spectral function  $\rho_{had}(s)$  is determined by the pion form factor  $F_\pi(s)$  for the energy interval  $4m_\pi^2 \div 0.81 \text{ GeV}^2$ :

$$\rho_{had}(s) = \frac{(s - 4m_\pi^2)^{3/2}}{12s^{5/2}} |F_\pi(s)|^2. \quad (51)$$

The contributions of resonances  $J^{PC} = 1^{--}$  of  $J/\psi$  and  $\Upsilon$  families and other nonresonance energy  $s$  intervals are calculated as in our previous works [22]. Total hadron vacuum polarization contribution is presented in Table I.

Six order  $\alpha$  corrections are also shown in Fig. 6. To evaluate the contribution of the amplitude in Fig. 6(a) we use an expansion of deuteron magnetic form factor:

$$G_M(k^2) = \frac{m_2}{Zm_p} \mu_d \left(1 - \frac{1}{6} r_M^2 \mathbf{k}^2\right), \quad (52)$$

which leads to the following potential in momentum space:

$$\Delta V^{hfs}(k) = -\frac{4\pi\alpha\mu_d}{9m_1m_p}r_M^2(\mathbf{s}_1\mathbf{s}_2)\mathbf{k}^2, \quad (53)$$

and in coordinate representation

$$\Delta V^{hfs}(k) = \frac{4\pi\alpha\mu_d}{9m_1m_p}r_M^2(\mathbf{s}_1\mathbf{s}_2)\nabla^2\delta(\mathbf{r}). \quad (54)$$

Averaging (54) over the Coulomb wave functions, we obtain an analytical expression for a correction to HFS and its numerical values for the levels  $1S$  and  $2S$ :

$$\Delta E_{1\gamma, str}^{hfs} = \frac{2}{3}\mu^2\alpha^2r_M^2\frac{3n^2+1}{n^2}E_F = \begin{cases} 1S : 0.0082 \text{ meV} \\ 2S : 0.0008 \text{ meV} \end{cases} \quad (55)$$

Another nuclear structure correction of order  $\alpha^6$  to HFS of muonic deuterium is determined in second order PT by the amplitude presented in Fig. 6(b). Each of the perturbation potentials in this diagram is proportional to  $\delta(\mathbf{r})$  if we use an expansion over small transfer momentum. As a result a contribution of second order PT is proportional to divergent expression  $\tilde{G}(0, 0)$ . To avoid an appearance of  $\tilde{G}(0, 0)$  we use the nuclear structure perturbation potential in the form:

$$\Delta V_{str, 1\gamma}^C(k) = -\frac{Z\alpha}{\mathbf{k}^2} \left[ \frac{1}{(1 + \frac{k^2}{\Lambda^2})^2} - 1 \right] = \frac{Z\alpha}{\Lambda^2} \frac{(2 + \frac{k^2}{\Lambda^2})}{(1 + \frac{k^2}{\Lambda^2})^2}, \quad \Lambda = \frac{\sqrt{12}}{r_d}. \quad (56)$$

A convenience of dipole parametrization for the deuteron charge form factor in this case instead of a parametrization from [21] is related with a fact that in the coordinate representation we obtain under such conditions sufficiently simple expressions:

$$\Delta V_{str, 1\gamma}^C(r) = \frac{Z\alpha(2 + \Lambda r)}{8\pi r} e^{-\Lambda r}. \quad (57)$$

Using the Green's functions (20) and (21) the analytical integration in second order PT can be performed. It gives the following result:

$$\Delta E_{str, SOPT}^{hfs}(1S) = E_F(1S) \frac{\mu\alpha}{2\pi\Lambda(1 + \frac{2W}{\Lambda})^4} \left\{ -2\frac{W}{\Lambda} \left[ 4\frac{W}{\Lambda}(5 + 3\frac{W}{\Lambda}) + 13 \right] - \right. \quad (58)$$

$$\left. -16\frac{W}{\Lambda} \left( 1 + \frac{W}{\Lambda} \right) \left( 1 + \frac{2W}{\Lambda} \right) \coth^{-1} \left( 1 + \frac{4W}{\Lambda} \right) - 3 \right\} = -0.0555 \text{ meV},$$

$$\Delta E_{str, SOPT}^{hfs}(2S) = -E_F(2S) \frac{\mu\alpha}{8\pi\Lambda(1 + \frac{W}{\Lambda})^6} \left\{ \frac{W}{\Lambda} \left[ \left( \frac{W}{\Lambda} \left( 14 + 3\frac{W}{\Lambda} \right) + 31 \right) \frac{W^2}{\Lambda^2} + 16 \right] + \right. \quad (59)$$

$$\left. + 8\frac{W}{\Lambda} \left( 1 + \frac{W}{\Lambda} \right) \left[ \left( 3 + \frac{W}{\Lambda} \right) \frac{W^2}{\Lambda^2} + 4 \right] \coth^{-1} \left( 1 + \frac{2W}{\Lambda} \right) + 6 \right\} = -0.0069 \text{ meV}.$$

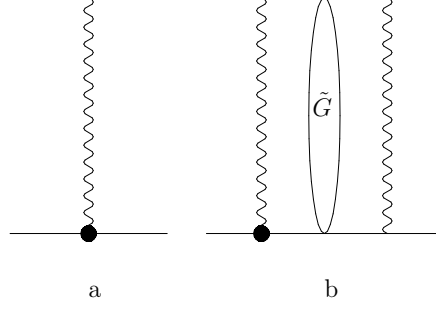


FIG. 6: Nuclear structure effects in one-photon interaction and in second order perturbation theory.  $\tilde{G}$  is the reduced Coulomb Green's function.

## V. RADIATIVE CORRECTIONS TO TWO PHOTON EXCHANGE DIAGRAMS.

The lepton line radiative corrections to two-photon exchange amplitudes are of order  $\alpha(Z\alpha)^5$ . Such corrections to HFS of muonium were studied in detail in [23]. Total integral expression for all radiative corrections in Fig. 7 to HFS of order  $\alpha(Z\alpha)^5$  including recoil effects was constructed in [24] in Fried-Yennie gauge [25]. The advantage of Fried-Yennie gauge in the calculation of corrections in Fig. 7 is that it leads to infrared finite renormalizable integral expressions for muon self-energy operator, vertex function and lepton tensor describing the diagram with spanning photon [26]. Using such general expressions an analytical calculation of corrections  $\alpha(Z\alpha)^5$  to HFS in the point-like nucleus approximation can be performed. If the effects of nuclear structure should be taken into account then these expressions allow us to obtain numerical values of diagrams in Fig. 7(a,b,c) separately. The muon-deuteron scattering amplitude corresponding to direct two-photon exchange diagrams with radiative insertions in muon line can be presented in the form:

$$\mathcal{M} = \frac{-i(Z\alpha)^2}{\pi^2} \int d^4k [\bar{u}(q_1)L_{\mu\nu}u(p_1)] D_{\mu\omega}(k)D_{\nu\lambda}(k) \times \quad (60)$$

$$[\epsilon_\rho^*(q_2)\Gamma_{\omega,\rho\beta}(q_2, p_2 + k)\mathcal{D}_{\beta\tau}(p_2 + k)\Gamma_{\lambda,\tau\alpha}(p_2 + k, p_2)\epsilon_\alpha(p_2)]$$

where  $\epsilon_\rho(q)$  is the wave function of free deuteron with spin 1,  $p_{1,2}$  and  $q_{1,2}$  are four-momenta of the muon and deuteron in initial and final states:  $p_{1,2} \approx q_{1,2}$ . The vertex operator describing the photon-deuteron interaction is determined by three form factors as follows:

$$\Gamma_{\omega,\rho\sigma}(q_2, p_2 + k) = \frac{(2p_2 + k)_\omega}{2m_2} g_{\rho\sigma} \cdot F_1(k) - \frac{(2p_2 + k)_\omega}{2m_2} \frac{k_\rho k_\sigma}{2m_2^2} \cdot F_2(k) - (g_{\rho\gamma}g_{\sigma\omega} - g_{\rho\omega}g_{\sigma\gamma}) \frac{k_\gamma}{2m_2} \cdot F_3(k). \quad (61)$$

The deuteron propagator and the photon propagator in the Coulomb gauge are equal to

$$\mathcal{D}_{\alpha\beta}(p) = \frac{-g_{\alpha\beta} + \frac{p_\alpha p_\beta}{m_2^2}}{(p^2 - m_2^2 + i0)}, \quad D_{\lambda\sigma}(k) = \frac{1}{k^2} \left[ g_{\lambda\sigma} + \frac{k_\lambda k_\sigma - k_0 k_\lambda g_{\sigma 0} - k_0 k_\sigma g_{\lambda 0}}{\mathbf{k}^2} \right]. \quad (62)$$

The lepton tensor  $L_{\mu\nu}$  has a definite form for three amplitudes in Fig. 7. Using the package FeynCalc [28] we perform independent construction of lepton tensors corresponding to muon self-energy, vertex corrections and the diagram with spanning photon:

$$L_{\mu\nu}^{se} = -\frac{3\alpha}{4\pi} \gamma_\mu (\hat{p}_1 - \hat{k}) \gamma_\nu \int_0^1 \frac{(1-x)dx}{(1-x)m_1^2 + x\mathbf{k}^2}, \quad (63)$$

$$L_{\mu\nu}^{vertex} = 2\frac{\alpha}{4\pi} \int_0^1 dz \int_0^1 dx \gamma_\mu \frac{\hat{p}_1 - \hat{k} + m_1}{(p_1 - k)^2 - m_1^2 + i0} \left[ F_\nu^{(1)} + \frac{F_\nu^{(2)}}{\Delta} + \frac{F_\nu^{(3)}}{\Delta^2} \right], \quad (64)$$

$$F_\nu^{(1)} = -6x\gamma_\nu \ln \frac{m_1^2 x + \mathbf{k}^2 z(1 - xz)}{m_1^2 x}, \quad F_\nu^{(3)} = 2x^3(1 - x)\hat{Q}(\hat{p}_1 - \hat{k} + m_1)\gamma_\nu(\hat{p}_1 + m_1)\hat{Q}, \quad (65)$$

$$F_\nu^{(2)} = -x^3(2\gamma_\nu Q^2 - 2\hat{Q}\gamma_\nu\hat{Q}) - x^2[\gamma_\alpha\hat{Q}\gamma_\nu(\hat{p}_1 + m_1)\gamma_\alpha + \gamma_\alpha(\hat{p}_1 - \hat{k} + m_1)\gamma_\nu\hat{Q}\gamma_\alpha + 2\gamma_\nu(\hat{p}_1 + m_1)\hat{Q} + 2\hat{Q}(\hat{p}_1 - \hat{k} + m_1)\gamma_\nu] - x(2 - x)\gamma_\alpha(\hat{p}_1 - \hat{k} + m_1)\gamma_\nu(\hat{p}_1 + m_1)\gamma_\alpha, \quad (66)$$

$$Q = -p_1 + kz, \quad \Delta = x^2 m_1^2 - xz(1 - xz)k^2 + 2kp_1 xz(1 - x),$$

$$L_{\mu\nu}^{jellyfish} = \frac{\alpha}{4\pi} \int_0^1 dz \int_0^1 dx \left( \frac{F_{\mu\nu}^{(1)}}{\Delta} + \frac{F_{\mu\nu}^{(2)}}{\Delta^2} + \frac{F_{\mu\nu}^{(3)}}{\Delta^3} \right). \quad (67)$$

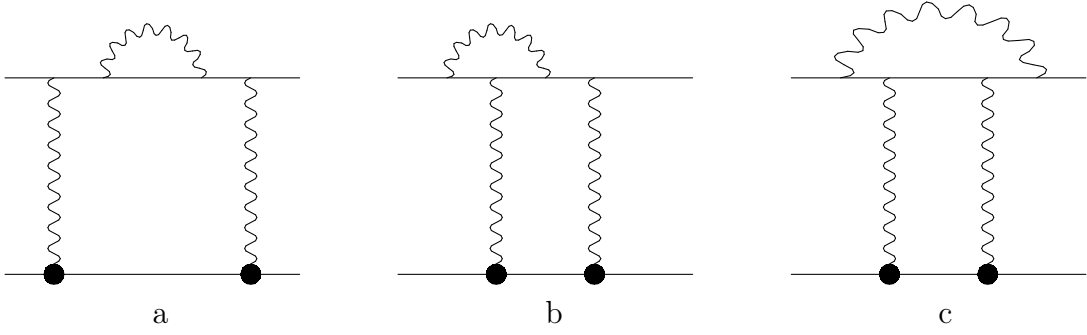


FIG. 7: Direct two-photon exchange amplitudes with radiative corrections to muon line giving contributions of order  $\alpha(Z\alpha)^5$  to the hyperfine structure. Wave line on the diagram denotes the photon. Bold point on the diagram denotes the vertex operator of the proton or deuteron.

Explicit form of tensors  $F_{\mu\nu}^{(i)}$  is presented in [27]. For a construction of hyperfine potential respective to the amplitude (60) we use the projection operators (41). The insertion (41) in (60) allows us to pass to the trace calculation and a contraction over the Lorentz indices by means of the system Form [20]. Hence, a general structure of potentials contributing to the energy shifts for states with the angular momenta 1/2 and 3/2 is the following:

$$N_{1/2} = \frac{1}{6} Tr \left\{ \sum_{\sigma} \Psi^{\sigma}(P) \bar{\Psi}^{\sigma}(P) (\gamma_{\rho} - v_{1,\rho}) \gamma_5 (1 + \hat{v}_1) L_{\mu\nu} (1 + \hat{v}_1) \gamma_5 (\gamma_{\alpha} - v_{1,\alpha}) \right\} \times \quad (68)$$

$$\Gamma_{\omega,\rho\beta}(q_2, p_2 + k) \mathcal{D}_{\beta\tau}(p_2 + k) \Gamma_{\lambda,\tau\alpha}(p_2 + k, p_2) D_{\mu\omega}(k) D_{\nu\lambda}(k),$$

$$N_{3/2} = \frac{1}{4} Tr \left\{ \sum_{\sigma} \Psi_{\alpha}^{\sigma}(P) \bar{\Psi}_{\rho}^{\sigma}(P) (1 + \hat{v}_1) L_{\mu\nu} (1 + \hat{v}_1) \right\} \times \quad (69)$$

$$\Gamma_{\omega,\rho\beta}(q_2, p_2 + k) \mathcal{D}_{\beta\tau}(p_2 + k) \Gamma_{\lambda,\tau\alpha}(p_2 + k, p_2) D_{\mu\omega}(k) D_{\nu\lambda}(k).$$

Neglecting the recoil effects in the denominator of the deuteron propagator we obtain:  $1/[(p_2 + k)^2 - m_2^2 + i0] \approx 1/(k^2 + 2kp_2 + i0) \approx 1/(2k_0 m_2 + i0)$ . The contribution of crossed two-photon exchange diagrams is also determined by (60)-(67) with a replacement  $k \rightarrow -k$  in the deuteron propagator. Then a summary contribution is proportional to  $\delta(k_0)$ :

$$\frac{1}{2m_2 k_0 + i0} + \frac{1}{-2m_2 k_0 + i0} = -\frac{i\pi}{m_2} \delta(k_0). \quad (70)$$

As a result three types of corrections of order  $\alpha(Z\alpha)^5$  to HFS of muonic deuterium are expressed in integral form over the loop momentum  $\mathbf{k}$  and the Feynman parameters:

$$\Delta E_{se}^{hfs} = E_F 6 \frac{\alpha(Z\alpha)}{\pi^2} \int_0^1 x dx \int_0^\infty \frac{F_1(k^2) F_3(k^2) dk}{x + (1-x)k^2}, \quad (71)$$

$$\Delta E_{vertex-1}^{hfs} = -E_F 24 \frac{\alpha(Z\alpha)}{\pi^2} \int_0^1 dz \int_0^1 x dx \int_0^\infty \frac{F_1(k^2) F_3(k^2) \ln[\frac{x+k^2 z(1-xz)}{x}] dk}{k^2}, \quad (72)$$

$$\Delta E_{vertex-2}^{hfs} = E_F 8 \frac{\alpha(Z\alpha)}{\pi^2} \int_0^1 dz \int_0^1 dx \int_0^\infty \frac{dk}{k^2} \left\{ \frac{F_1(k^2) F_3(k^2)}{[x + k^2 z(1-xz)]^2} \left[ -2xz^2(1-xz)k^4 + \right. \right. \quad (73)$$

$$\left. \left. zk^2(3x^3z - x^2(9z+1) + x(4z+7) - 4) + x^2(5-x) \right] - \frac{1}{2} \right\},$$

$$\Delta E_{jellyfish}^{hfs} = E_F 4 \frac{\alpha(Z\alpha)}{\pi^2} \int_0^1 (1-z) dz \int_0^1 (1-x) dx \int_0^\infty \frac{F_1(k^2) F_3(k^2) dk}{[x + (1-x)k^2]^3} \quad (74)$$

$$\times \left[ 6x + 6x^2 - 6x^2z + 2x^3 - 12x^3z - 12x^4z + k^2(-6z + 18xz + 4xz^2 + 7x^2z - 30x^2z^2 - \right.$$

$$\left. 2x^2z^3 - 36x^3z^2 + 12x^3z^3 + 24x^4z^3) + k^4(9xz^2 - 31x^2z^3 + 34x^3z^4 - 12x^4z^5) \right]$$

The contribution of form factor  $F_2(k^2)$  in (71)-(74) is omitted because the terms  $F_2(k^2)F_3(k^2)$  are suppressed by powers of the mass  $m_2$ . The term  $1/2$  in figure brackets (73) is related to the subtraction term of the quasipotential. All corrections (71), (72), (73), (74) are expressed through the convergent integrals. It is necessary to point out that expressions for the vertex function and lepton tensor with spanning photon were obtained in [26] in a slightly different form. Nevertheless, in the case of point-like nucleus they lead to contributions to hyperfine structure of  $S$ -states which coincide with (71)-(74). In numerical evaluation of finite size corrections (71)-(74) we use the deuteron form factor parametrization from [21]. Numerical results are presented in Table I.

## VI. CONCLUSION

In this work we calculate QED corrections, nuclear structure and recoil corrections of orders  $\alpha^5$  and  $\alpha^6$  to hyperfine splitting of  $1S$  and  $2S$  energy levels in muonic deuterium. In contrast to earlier performed investigations of the energy spectra of light muonic atoms in [3] we use three-dimensional quasipotential approach for the description of the muon-deuteron bound state. All considered corrections due to effects of vacuum polarization and deuteron structure are presented in integral form and calculated numerically. Numerical values of studied corrections are exhibited in Table I. We present in Table I relevant references on equations which allow to analyze again the sources of corresponding corrections. In line 5 of Table I we present a sum of corrections of order  $\alpha^6$  which includes two-loop electron vacuum polarization corrections in first and second orders of perturbation theory and muon one-loop vacuum polarization corrections in first and second order of perturbation theory.

As pointed out above the hyperfine structure of muonic atoms was investigated in [3, 4]. In these papers the transition frequencies between the energy levels  $2P$  and  $2S$  were obtained

in the case of muonic hydrogen, muonic deuterium and ions of muonic helium. The only detailed calculation of  $2S$ -state hyperfine splitting in muonic deuterium is presented in [3]. The splitting formula obtained in this paper

$$\Delta E_{2s} = \frac{3}{2}\beta_D(1 + a_\mu)(1 + \epsilon_{VP} + \epsilon_{vertex} + \epsilon_{Breit} + \epsilon_{Zemach}) = 6.0584(7) \text{ meV} \quad (75)$$

contains basic corrections to the Fermi energy: the vacuum polarization, relativistic correction, vertex correction and the Zemach contribution. Note that the Zemach correction  $(-0.1177(7))$  meV for  $2S$ -state in muonic deuterium from [3] is slightly different from our value  $(-0.1163)$  meV what may be related with recoil effects. One-loop electron vacuum polarization corrections in first order  $\epsilon_{VP1} = 0.00218$  and in second order PT  $\epsilon_{VP2} = 0.00337$  for  $2S$ -state from [3] coincide exactly with our results expressed by (12) and (23). As it follows from our Table I the total value of  $2S$ -state hyperfine splitting  $6.0683$  meV without an account of deuteron polarizability contribution obtained in [5] is in good agreement with the result  $6.0584$  meV obtained in [3]. A small difference in the results is conditioned first of all by nuclear recoil, structure and polarizability effects in two-photon exchange diagrams which we calculate using modern experimental data on deuteron electromagnetic form factors, and by nuclear structure corrections of order  $\alpha^6$ . We include in Table I numerical value of deuteron polarizability contribution to hyperfine splitting in muonic deuterium evaluated by means of analytical expression obtained in [5] for electronic deuterium in the zero range approximation. This is main part of polarizability correction. The estimate of internal deuteron polarizability correction is made in Table I on the basis of results for muonic hydrogen. This contribution is now under consideration as in our work [19].

Assuming that the deuteron electromagnetic form factor parameterizations were obtained with an uncertainty near 0.5 per cent at small values of photon momentum squared  $Q^2$  we obtain that theoretical error in the calculation of basic nuclear structure contribution of order  $(Z\alpha)^5$  which is determined by a product of two electromagnetic form factors can not be less than one per cent or  $\pm 0.0010$  meV for  $2S$ -state. Another source of the error is related with recoil effects of order  $\alpha(Z\alpha)^5 m_1/m_2$  in amplitudes in Fig. 7, which can amount the value  $0.00002$  meV. The error in determination of internal polarizability correction is estimated approximately on muonic hydrogen in  $0.0025$  meV ( $2S$ -state) (near 25 per cent). The estimate of an uncertainty in deuteron polarizability correction is given on the basis of results from [5]. It is equal approximately to  $0.0042$  meV (2 per cent). Weak interaction contribution is equal to zero in nonrelativistic approximation as was demonstrated in [8, 30]. Our total theoretical error is estimated in  $0.0050$  meV in the case of  $2S$ -state. To obtain this estimate we add the above mentioned uncertainties in quadrature. The hyperfine structure interval  $\Delta_{12}$  does not contain uncertainties with nuclear structure and polarizability. So, the obtained in this work value  $\Delta_{12} = -0.0379$  meV can be used for additional check of quantum electrodynamics in the case of muonic deuterium with a precision  $0.001$  meV. To construct the quasipotential corresponding to amplitudes in Fig. 4 we develop the method of projection operators on the bound states with definite spins. It allows to employ different systems of analytical calculations [20, 28]. In this approach more complicated corrections, for example, radiative recoil corrections to hyperfine structure of order  $\alpha(Z\alpha)^5 m_1/m_2$  can be evaluated if an increase of the accuracy will be needed. The results from Table I should be taken into account for a comparison with experimental data [1].

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